Lecture 5

- Recap
- Plane waves (cont.)
- Radiation of a moving charge: Lienard - Wiechert potentials

Recap:

$$G_R = \frac{\delta\left(t - t' - \frac{r}{c}\right)}{4\pi r} \qquad \Box G_R = \delta^3(\vec{x} - \vec{x}') \cdot \delta(t - t')$$

$$u = \frac{\varepsilon_0}{2}|E^2| + \frac{1}{2\mu_0}|B^2| \rightarrow \text{EM Energy}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} \rightarrow \text{Energy Flux}$$

$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$

$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

$\nabla \times E = -\frac{\partial B}{\partial t}$

$\nabla \cdot B = 0$

$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t}$

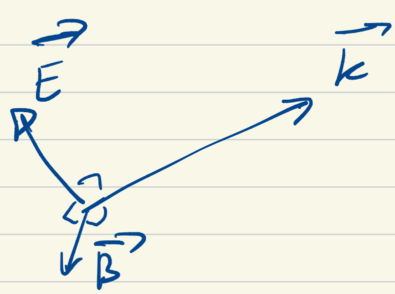
$\Box \varphi = \frac{\rho}{\varepsilon_0}$

$\Box A = \mu_0 \vec{J}$

- Plane waves (continuation)

$\vec{E} = \hat{\vec{E}}(k) e^{i\vec{k}\cdot\vec{x} - i\omega t} + c.c.$

$\vec{B} = \hat{\vec{B}}(k) e^{i\vec{k}\cdot\vec{x} - i\omega t} + c.c.$



Derivation below:

$\hat{\vec{E}} = ic|k|\vec{A}_\perp \qquad \hat{B} = i\vec{k} \times \hat{\vec{A}}_\perp \qquad \vec{B} = \frac{1}{c}\vec{n} \times \vec{E}$

$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{1}{c\mu_0}\vec{E} \times \vec{n} \times \vec{E} = \frac{c}{\varepsilon_0}|E|^2 = cu$

$$\hat{\Phi}(x,t) = \hat{\varphi}(\vec{k})\, e^{-i\varphi}$$

$$\hat{A}(x,t) = \hat{A}(\vec{k})\, e^{-i\varphi}$$

$$\varphi = \omega_k t - \vec{k}\cdot\vec{x}$$

$$\omega_k = c\,|\vec{k}|$$

$$\vec{\partial}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$$

$$\hat{\varphi} = \frac{c^2}{\omega_k}\vec{k}\cdot\hat{\vec{A}} = \frac{c\vec{k}\cdot\vec{A}}{|k|}$$

Now let us find $\vec{E}$ and $\vec{B}$

We use a similar Ansatz:

$\varphi = i\omega t - ikx$

$$\vec{E} = \hat{\vec{E}}(k)\,e^{-i\varphi} + c.c.$$

$$\vec{B} = \hat{\vec{B}}(k)\,e^{-i\varphi} + c.c.$$

$$\vec{E} = -\vec{\partial}\varphi - \dot{\vec{A}} = \left(i\vec{k}\hat{\varphi} - i\omega_k\hat{A}\right)e^{-i\varphi} + c.c.$$

$$\Rightarrow \hat{E} = -ic\frac{\vec{k}}{|k|}\,\vec{k}\cdot\vec{A} + ic|k|\vec{A} \equiv ick\hat{A}_\perp$$

$$\vec{B} = \vec{\partial}\times\vec{A} \Rightarrow \hat{\vec{B}}_\alpha = i\varepsilon_{\alpha\beta\gamma}k_\beta(\hat{A}_\perp)_\gamma$$

Lienard - Wiechert potentials.

We will find the EM fields produced by a point-like charge q moving along a trajectory $x_0(t)$. This corresponds to the charge density and the current:

$$\rho(\vec{x},t) = q\,\delta^3(\vec{x} - \vec{x}_0(t))$$

$$\vec{J}(\vec{x},t) = q\,\vec{v}(t)\,\delta^3(\vec{x} - \vec{x}_0(t))$$

First, check conservation:

$$\dot{\rho} + \vec{\nabla}\cdot\vec{J} = -q\left(\vec{\nabla}\delta^3(\vec{x} - \vec{x}_0(t))\cdot\dot{\vec{x}}_0(t)\right)$$

$$+\,q\vec{v}(t)\cdot\vec{\nabla}\delta^3(\vec{x} - \vec{x}_0(t)) = 0$$

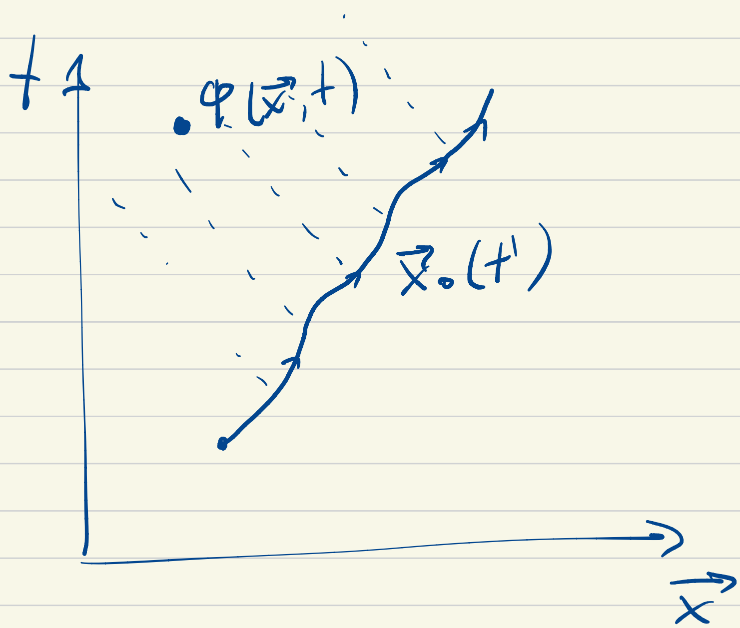
We will determine φ and $\vec{A}$ and from them $\vec{E}$ and $\vec{B}$.

For a charge at a given moment in time the pot. is given by the Greens function of the □. Hence for all times we need to integrate it over time:

$$\varphi(x,t) = \frac{q}{4\pi\varepsilon_0}\int dt'\, G_{ret.}(\vec{x},t,\vec{x}_0(t'),t') =$$

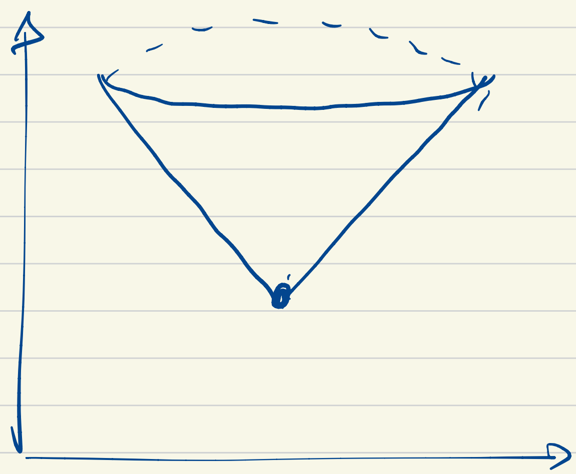
$$= \frac{q}{4\pi\varepsilon_0}\int dt'\, \frac{\delta(t-t'-|\vec{x}-\vec{x}_0(t')|\cdot\frac{1}{c})}{|\vec{x}-\vec{x}_0(t')|}$$

and similar for $\vec{A}$.



It is natural to use Gret because we do not want to consider the field that "existed before the charge."

Green's function is localized on the "light-cone" [only in odd dimensions]



$$c(t-t') = |\vec{x} - \vec{x}_0|$$

It means that there is one and only one solution of

$$f(t') = t - t' - \frac{|\vec{x} - \vec{x}_0(t')|}{c} = 0$$

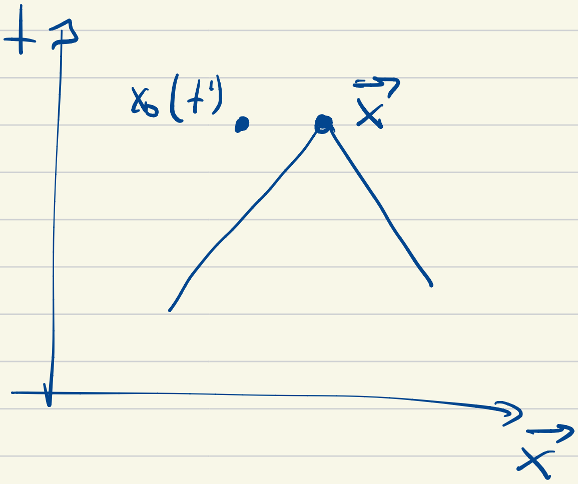
if $v < c$:

$$\frac{df}{dt'} = -1 + \frac{1}{c}\vec{n}\cdot\vec{v}(t') < 0$$ (not more than one solution)

$$\vec{n} = \frac{\vec{R}}{|R|} \qquad \vec{R} = \vec{x} - \vec{x}_0(t')$$

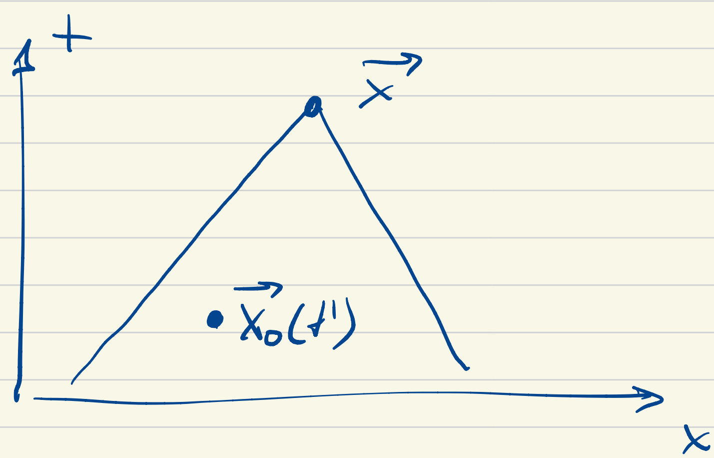
Let's show that solution exists:

for $t' = t$ $\quad g(t') \leq 0$:



for $t' \ll t$ $\quad g(t') \geq 0$, because

$\vec{x}_0(t')$ is "inside the past lightcone of the point $\vec{x}, t$:



light cone: $\quad |\vec{x}' - \vec{x}| = c(t - t') \quad [g = 0]$

By continuity trajectory $\vec{x}_0(t')$ crosses the lightcone.

We will denote solution of $f(t')=0$ as t_*. It is a function of t and $\vec{x}$ that we keep fixed.

$$t_*(t,\vec{x}) \qquad \text{[not finding it explicitly]}$$

Now
$$\int dz\, \delta(f(z))\, F(z) = \frac{F(z_*)}{|f'(z_*)|}$$

$$z_*: \ f(z_*)=0$$

Hence
$$\vec{\beta} = \frac{\vec{v}}{c}$$

$$\varphi(\vec{x},t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{|R(t_*)|\,(1-\vec{n}(t_*)\cdot\vec{\beta}(t_*))}$$

everything at $t'=t_*$!

$$\vec{A}(\vec{x},t) = \frac{q}{4\pi\varepsilon_0 c} \frac{\vec{\beta}}{|R|\,(1-\vec{n}\vec{\beta})} = \frac{q}{4\pi\varepsilon_0 c} \frac{\vec{\beta}}{|R|-\vec{R}\cdot\vec{\beta}}$$

$$\frac{1}{\mu_0\varepsilon_0}$$

$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$, to evaluate it we need to know derivatives of t_* wrt t and $\vec{x}$.

$$t - t_* - \frac{|\vec{x} - \vec{x}_0(t_*)|}{c} = 0$$

$$\frac{\partial}{\partial t}: \quad 1 - \frac{\partial t_*}{\partial t} + \frac{\vec{V}}{c} \cdot \frac{\vec{R}}{|R|} \frac{\partial t_*}{\partial t} = 0 \Rightarrow$$

$$\frac{\partial t_*}{\partial t} = \frac{1}{1 - \vec{\beta} \cdot \vec{n}} \qquad \frac{\partial}{\partial t}|R| = -\frac{\vec{V} \cdot \vec{R}}{|R|} \frac{\partial t_*}{\partial t}$$

$$\vec{\nabla}_x: \quad -\vec{\nabla} t_* - \frac{\vec{R}}{|R|c} + \frac{\vec{V}}{c} \cdot \frac{\vec{R}}{|R|} \vec{\nabla} t_* = 0 \Rightarrow$$

$$\vec{\nabla}|R| = \frac{\vec{R}}{|R|} - \frac{\vec{V} \cdot \vec{R}}{|R|} \vec{\nabla} t_*$$

$$\vec{\nabla} t_* = -\frac{\vec{n}}{c}\left(\frac{1}{1 - \vec{\beta} \cdot \vec{n}}\right)$$

$$\vec{R} = \vec{x} - \vec{x}_0(t_*)$$

$$\vec{\nabla} \frac{1}{|R| - \vec{R}\cdot\vec{\beta}} = \frac{-1}{\left(|R| - \vec{R}\cdot\vec{\beta}\right)^2} \left[\frac{\vec{R}}{|R|} + \frac{\vec{v}\cdot\vec{n}}{c} \vec{n} \left(\frac{1}{1-\vec{\beta}\cdot\vec{n}} \right) - \right.$$

$$\left. - \vec{\beta} - (\vec{v}\cdot\vec{\beta}) \frac{\vec{n}}{c} \frac{1}{1-\vec{\beta}\cdot\vec{n}} + (\vec{R}\cdot\dot{\vec{\beta}}) \cdot \frac{\vec{n}}{c} \left(\frac{1}{1-\vec{\beta}\cdot\vec{n}} \right) \right] =$$

$$= - \frac{1}{\left(|R| - \vec{R}\cdot\vec{\beta}\right)^2} \left[\vec{n} \left(1 + \frac{\vec{\beta}\cdot\vec{n} - \beta^2 + (\vec{R}\cdot\dot{\vec{\beta}}) \frac{1}{c}}{1-\vec{\beta}\cdot\vec{n}} \right) - \vec{\beta} \right]$$

$$= - \frac{1}{\left(|R| - \vec{R}\cdot\vec{\beta}\right)^2} \left[\frac{\vec{n}\left(1 - \beta^2 + \vec{R}\cdot\dot{\vec{\beta}} \frac{1}{c}\right)}{1-\vec{\beta}\cdot\vec{n}} - \vec{\beta} \right]$$

$$\frac{\partial}{\partial t} \frac{\vec{\beta}}{|R| - \vec{R}\cdot\vec{\beta}} = \frac{\dot{\vec{\beta}} \cdot |R|}{\left(|R| - \vec{R}\cdot\vec{\beta}\right)^2} - \frac{\vec{\beta}}{\left(|R| - \vec{R}\cdot\vec{\beta}\right)^2} \cdot$$

$$\cdot \left[- \frac{\vec{v}\cdot\vec{R}}{|R|} \frac{1}{1-\vec{\beta}\cdot\vec{n}} + \frac{\vec{v}\cdot\vec{\beta}}{1-\vec{\beta}\cdot\vec{n}} - \frac{\vec{R}\cdot\dot{\vec{\beta}}}{1-\vec{\beta}\cdot\vec{n}} \right]$$

$$\vec{E} = -\frac{q}{4\pi\varepsilon_0}\left(\vec{\nabla}\cdot\frac{1}{|R| - \vec{R}\cdot\vec{\beta}} + \frac{1}{c}\frac{\partial}{\partial t}\frac{\vec{\beta}}{|R| - \vec{R}\cdot\vec{\beta}}\right)$$

Look for $\sim \frac{1}{R^2}$ terms:

$$\frac{q}{4\pi\varepsilon_0}\frac{1}{(|R| - \vec{R}\cdot\vec{\beta})^2}\left(\frac{\vec{n}(1-\beta^2)}{1-\vec{\beta}\cdot\vec{n}} - \frac{\vec{\beta}\left[(1-\vec{\beta}\cdot\vec{n}) - \beta^2 + \vec{\beta}\cdot\vec{n}\right]}{(1-\vec{\beta}\cdot\vec{n})}\right)$$

$$= \frac{q}{4\pi\varepsilon_0 |R^2|}\frac{(1-\beta^2)}{(1-\vec{\beta}\cdot\vec{n})^3}(\vec{n} - \vec{\beta})$$

Look for $\sim \frac{1}{R}$ terms:

$$\frac{q}{4\pi\varepsilon_0}\frac{1}{(|R| - \vec{R}\cdot\vec{\beta})^2}\left[\frac{\vec{n}}{c}\frac{(\vec{R}\cdot\dot{\vec{\beta}})}{1-\vec{\beta}\cdot\vec{n}} - \frac{\dot{\vec{\beta}}}{c}|R| - \frac{\vec{\beta}}{c}\frac{\vec{R}\cdot\dot{\vec{\beta}}}{1-\vec{\beta}\cdot\vec{n}}\right] =$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{(1-\vec{\beta}\cdot\vec{n})^3} \frac{1}{cR} \left[(\vec{n}\cdot\dot{\vec{\beta}})(\vec{n}-\vec{\beta}) - \dot{\vec{\beta}}(1-\vec{\beta}\vec{n}) \right]$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{(1-\vec{\beta}\cdot\vec{n})^3} \frac{1}{cR} \vec{n} \times \left[\vec{n}-\vec{\beta} \right] \times \dot{\vec{\beta}}$$

[all at t_*]

$$\left[a \times (b \times \vec{c}) = \vec{b}\cdot(\vec{a}\cdot\vec{c}) - \vec{c}\cdot(\vec{a}\cdot\vec{b}) \right]$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \ldots = \frac{1}{R} \left[\vec{R} \times \vec{E} \right] \qquad \vec{B} \perp \vec{E}$$

analogous!

- Let's study the $\frac{1}{R^2}$ term:

$$\frac{q}{4\pi\varepsilon_0 |\vec{R}^2|} \frac{(1-\beta^2)}{(1-\vec{\beta}\cdot\vec{n})^3} (\vec{n}-\vec{\beta})$$

it is independent of the acceleration –

– it is the Coulomb field of the

moving charge: $|\vec{R} - \beta \cdot \vec{R}|$ → is the distance to the charge at moment t (for constant velocity).

various β - dependent factors come from "changing the frame".

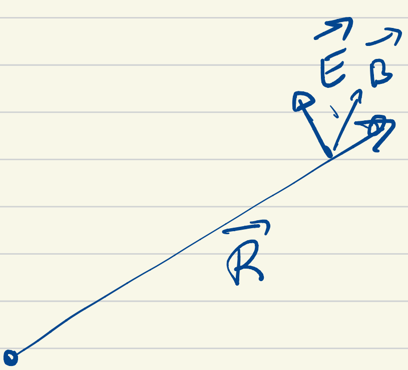
- The $\frac{1}{R}$ term is the radiation

$$\frac{q}{4\pi\varepsilon_0} \frac{1}{(1-\vec{\beta}\cdot\vec{n})^3} \frac{1}{cR} \vec{n} \times [\vec{n}-\vec{\beta}] \times \dot{\vec{\beta}}$$

the energy flux:

$$\frac{d\mathcal{E}}{d\Omega\, dt} = \lim_{R\to\infty} R^2 \vec{S}\cdot\vec{n} \,, \qquad \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$\frac{1}{\mu_0}\left(\vec{E} \times \vec{n} \times \vec{E}\right)\cdot\vec{n} = \frac{E^2}{\mu_0}$$



$$\frac{1}{c\mu_0} \lim_{R\to\infty} R^2 |\vec{E}|^2 =$$

$$= \frac{q^2}{16\pi^2 \underbrace{\varepsilon_0^2 \mu_0 c^3}_{\varepsilon_0 c}} \frac{\left(\vec{n} \times (\vec{n}-\vec{\beta}) \times \dot{\vec{\beta}}\right)^2}{(1-\vec{n}\cdot\vec{\beta})^6}$$

• Non-relativistic limit:

$|\vec{\beta}| \ll 1 \qquad \left[\vec{\beta} = \frac{\vec{v}}{c} \right] \qquad \dot{\vec{\beta}}$

$$\frac{d\mathcal{E}}{d\Omega\, dt} = \frac{q^2}{16\pi^2 \varepsilon_0 c} \left| \vec{n} \times \dot{\vec{\beta}} \right|^2$$

$$\frac{dP}{dt} = \int d\Omega \sin^2\theta \frac{q^2 |\dot{\vec{\beta}}|^2}{16\pi^2 \varepsilon_0 c} =$$

$$\int_0^\pi d\varphi \sin\theta\, d\theta$$

$$\left[\int_0^\pi \sin^3\theta = \frac{4}{3} \right] \qquad = \frac{q^2 |\dot{\vec{\beta}}|^2}{6\pi^2 \varepsilon_0 c}$$

Larmor Formula

In the relativistic case it is important to distinguish the radiation emitted in the frame of the particle:

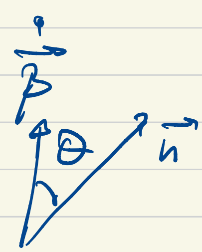
$$\frac{d\mathcal{E}}{d\Omega\, dt'} = \frac{d\mathcal{E}}{d\Omega\, dt} \cdot \frac{dt}{dt'} \qquad \frac{dt'}{dt} = \frac{1}{1-\vec{\beta}\cdot\vec{n}}$$

$$|\vec{\beta}| \simeq 1 \qquad 1-\vec{\beta}\cdot\vec{n} \longrightarrow 0$$

$$\vec{\beta} \parallel \vec{n}$$

- Rectilinear motion:

$$\beta \parallel \dot{\beta} \quad , \quad \frac{q^2}{16\pi^2\varepsilon_0 c}\,\frac{\left(\vec{n}\times(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)^2}{(1-\vec{n}\cdot\vec{\beta})^6}$$



$$\frac{d\mathcal{E}}{d\Omega\, dt'} = \frac{q^2}{16\pi^2\varepsilon_0 c}\,\frac{\dot{\vec{\beta}}^{\,2}\sin^2\Theta}{\left(1-\beta\cos\Theta\right)^5}$$

total power: $$\frac{d\mathcal{E}}{dt'} = \frac{q^2|\dot{\beta}|^2}{6\pi\varepsilon_0 c}\,\gamma^6 \;, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Derivation:

$$\int_0^{\pi} \frac{d\theta' \sin^3\theta}{(1-\beta\cos\theta)^5} = \int_{-1}^{1} \frac{dx\,(1-x^2)}{(1-\beta x)^5} = \frac{4}{3}\frac{1}{(1-\beta^2)^3}$$

$\beta \to 1$. $\gamma \to \infty$, all radiation near $\theta \sim 0$

$$\gamma \sim \frac{1}{\sqrt{2(1-\beta)}} \; : \; 2\gamma^2 \sim \frac{1}{1-\beta}$$

$$\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \to \frac{\theta^2}{\left(1-\beta\left(1-\frac{\theta^2}{2}\right)\right)} =$$

$$= \frac{\theta^2}{\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^5} = \frac{2^5\gamma^{10}\theta^2}{(1+\gamma^2\theta^2)^5}$$

$$\frac{d\mathcal{E}}{d\Omega\,dt'} = \frac{2q^2}{\pi^2\varepsilon_0 c}|\dot{\vec{\beta}}|^2\gamma^8\frac{(\gamma\theta)^2}{(1+(\theta\gamma)^2)^5}$$

$(\theta\gamma)^2 = \frac{1}{4} \to$ max.